

Delzant models of moduli spaces

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23.05.2001

Abstract

For every genus g , we construct a smooth, complete, rational polarized algebraic variety DM_g together with a normal crossing divisor $D = \bigcup D_i$, such that for every moduli space $M_\Sigma(2, 0)$ of semistable topologically trivial vector bundles of rank 2 on an algebraic curve Σ of genus g there exists a holomorphic isomorphism

$$f: M_\Sigma(2, 0) \setminus K_2 \rightarrow DM_g \setminus D,$$

where K_2 is the Kummer variety of the Jacobian of Σ , sending the polarization of DM_g to the theta divisor of the moduli space. This isomorphism induces isomorphisms of the spaces $H^0(M_\Sigma(2, 0), \Theta^k) = H^0(DM_g, H^k)$.

1 Introduction

At the last meeting of GAEL,¹ Bill Oxbury asked for a “topological” identification of $M_\Sigma(2, 0)$ with complex projective 3-space \mathbb{CP}^3 for any curve Σ of genus 2. To understand the problem, recall that, as a real manifold, this moduli space is the space $\text{ClRep}(\pi_1(\Sigma))$ of representations classes of the fundamental group $\pi_1(\Sigma)$ in $\text{SU}(2)$. The problem is to recognize \mathbb{CP}^3 in terms of this space.

By standard arguments of algebraic geometry, a complex structure on a compact Riemann surface Σ of genus 2 induces a complex structure on

¹Géométrie algébrique en liberté (9th edition, 19th–23rd March 2001), school organised by EAGER, EU project Contract No. HPRN-CT-2000-00099.

$\mathrm{ClRep}(\pi_1(\Sigma))$, as the moduli space $M_\Sigma(2, 0)$ of semistable rank 2 holomorphic vector bundles with trivial determinant. With this complex structure $\mathrm{ClRep}(\pi_1(\Sigma))$ is precisely \mathbb{CP}^3 . But we want to identify $\mathrm{ClRep}(\pi_1(\Sigma))$ directly with projective 3-space.

In particular, we claim the following:

- (1) as an algebraic variety, the moduli space $M_\Sigma(2, 0)$ is independent of Σ ;
- (2) it is rational;
- (3) the spaces of conformal blocks are independent of the moduli of the curve.

The space $\mathrm{ClRep}(\pi_1(\Sigma))$ of representation classes of $\pi_1(\Sigma)$ in $\mathrm{SU}(2)$ admits a symplectic form Ω that is canonical and defined purely topologically (see [2]). Thus symplectic arguments should be applicable, as should arguments from the theory of Hamiltonian torus actions or *symplectic toric geometry* (see the fundamental monograph [8], our main reference for technical details).

Recall the set-up of the theory of toric manifolds: let (M, ω) be a symplectic manifold of dimension $2n$ with a smooth Hamiltonian action of the n -dimensional torus T^n . That is, there is a map $f: T^n \rightarrow \mathrm{Diff} M$ preserving ω . Then action-angle coordinates define the moment map

$$\pi: M \rightarrow \Delta \subset \mathbb{R}^n, \quad (1.1)$$

whose image Δ is a convex polyhedron in Euclidean n -space. This polyhedron contains complete information on the symplectic geometry of (M, ω) . That is, Δ determines the manifold, the symplectic form and the T^n -action (see [1]).

Moreover if (M, ω) is prequantized (see [8]), and M has a Hodge structure whose Kähler form is ω , then this Hodge structure can also be reconstructed. The space $\mathrm{ClRep}(\pi_1(\Sigma))$ admits a well known Hamiltonian action of T^{3g-3} (see for example [7]). The differences in properties seem at first sight to be very slight:

- (1) for $g > 2$ the representation space $\mathrm{ClRep}(\pi_1(\Sigma))$ is singular, with singular locus the Kummer space

$$\mathrm{Sing} \mathrm{ClRep}(\pi_1(\Sigma)) = K_g = R_g^{\mathrm{U}(1)} / \pm \mathrm{id} \quad (1.2)$$

i.e., the space of $\mathrm{U}(1)$ -representations of $\pi_1(C)$ up to $\pm \mathrm{id}$;

- (2) our $3g - 3$ -torus action is only smooth over interior points of Δ , but continuous everywhere.

For example, in the case $g = 2$, the space $\text{ClRep}(\pi_1(\Sigma))$ admits an action of T^3 whose moment map Δ has image the tetrahedron

$$0 \leq t_i \leq 1, \quad |t_1 - t_2| \leq t_3 \leq \min(t_1 + t_2, 2 - t_1 - t_2) \quad (1.3)$$

in Euclidean space \mathbb{R}^3 with coordinates t_1, t_2, t_3 . This is a Delzant tetrahedron (see [8]), and it uniquely determines the Hodge Delzant variety DM_2 , which is just \mathbb{CP}^3 with the coordinate hyperplanes $\bigcup \mathbb{CP}_i^2$ for $i = 0, 1, 2, 3$ as distinguished divisor, that is, 4 planes in general position. We call it the Delzant model of $\text{ClRep}(\pi_1(\Sigma))$.

Using the complex structure on $\text{ClRep}(\pi_1(\Sigma))$ given by a complex structure on Σ , the equivariant Darboux–Weinstein theorem gives the holomorphic map

$$f: DM_2 \setminus \bigcup_{i=0}^4 \mathbb{CP}_i^2 \rightarrow M_\Sigma(2, 0) \setminus K_2. \quad (1.4)$$

Although $DM_2 = \mathbb{CP}^3 = M_\Sigma(2, 0)$ as rational algebraic varieties, the biholomorphic map f (1.4) cannot be extended to a biholomorphic identification $\mathbb{CP}^3 = DM_2 = M_\Sigma(2, 0)$. Instead, we turn to birational (symplectic) geometry.

Our aim here is to construct a Delzant model for any genus with the properties described in the abstract. The case of genus 2 prompts the way for this. Our construction gives in addition a finite chain of elementary “birational” transformations (flips) sending the Delzant model DM_g to the rational variety $(\mathbb{CP}^3)^{g-1}$ just as for toric varieties in algebraic geometry (see [5]).

The idea of constructing Delzant models comes from Donaldson [5], where a close cousin of DM_g was constructed for the smooth case $M_\Sigma(2, 1)$ by imitating a moduli space, to explain the appearance of Bernoulli numbers in the Verlinde formula.

2 Toric structures on $\text{ClRep}(\pi_1(\Sigma))$

Let Σ be a Riemann surface of genus g with fundamental group $\pi_1(\Sigma)$, and let C be a simple closed curve on Σ . We have the so-called *Goldman function*

on $\text{ClRep}(\pi_1(\Sigma))$:

$$c_C: \text{ClRep}(\pi_1(\Sigma)) \rightarrow [0, 1] \subset \mathbb{R}, \quad (2.1)$$

that sends a representative $\rho \in \text{ClRep}(\pi_1(\Sigma))$ to

$$\frac{1}{\pi} \cdot \cos^{-1} \left(\frac{1}{2} \text{Tr}(\rho([C])) \right) \in [0, 1] \quad (2.2)$$

where $[C]$ is the homotopy class of C . Goldman [2] proved that c_C is a Hamiltonian function of a $U(1)$ -action on $\text{ClRep}(\pi_1(\Sigma))$ for the canonical symplectic structure Ω . An exact formula for this action in simple geometric terms is given in [3]. Moreover, if C_1 and C_2 are two disjoint curves then

$$\{c_{C_1}, c_{C_2}\} = 0, \quad (2.3)$$

where the bracket is again with respect to Ω ; if $[C_1] \neq [C_2]$ then we obtain a Hamiltonian action of $T^2 = U(1) \times U(1)$ on $\text{ClRep}(\pi_1(\Sigma))$, and so on.

It is well known that a maximal set of disjoint inequivalent curves consists of $3g - 3$ curves. Fix one such set

$$\{C_1, \dots, C_{3g-3}\}. \quad (2.4)$$

The isotopy class of such a set of circles is called a *marking* of the Riemann surface. It is easy to see that the complement is the union

$$\Sigma_g \setminus \{C_1, \dots, C_{3g-3}\} = \coprod_{i=1}^{2g-2} P_i \quad (2.5)$$

of $2g - 2$ trinions P_i , where each *trinion* is a 2-sphere with 3 disjoint discs deleted:

$$P_i = S^2 \setminus (D_1 \cup D_2 \cup D_3). \quad (2.6)$$

On the other hand, any trinion decomposition of Σ is given by a choice of a maximal set of disjoint, noncontractible, pairwise nonisotopic smooth circles on Σ . It is easy to see that any such set consists of $3g - 3$ simple closed circles $C_1, \dots, C_{3g-3} \subset \Sigma_g$ with complement the union of $2g - 2$ trinions P_j . The type of such a decomposition is given by its *trivalent dual graph* $\Gamma(\{C_i\})$,

associating a vertex to each trinion P_i and an edge linking P_i and P_j to a circle C_l such that

$$C_l \subset \partial P_i \cap \partial P_j.$$

Thus the isotopy class of a trinion decomposition is given by a trivalent graph Γ .

On the other hand any trivalent graph Γ with set of vertices $V(\Gamma)$ and set of edges $E(\Gamma)$ defines a handlebody $\tilde{\Gamma}$, that is, a 3-manifold with boundary $\partial\tilde{\Gamma} = \Sigma_\Gamma$ (a Riemann surface of genus g with a trinion decomposition) by the “pumping trick” (see [4]): pump up the edges of Γ to tubes and the vertices to small 2-spheres. We get a Riemann surface Σ_Γ of genus g with a tube \tilde{e} for every $e \in E(\Gamma)$ and a trinion \tilde{v} for every $v \in V(\Gamma)$. The isotopy classes of meridian circles of tubes define $3g - 3$ disjoint, noncontractible, pairwise nonisotopic circles $\{C_e\}$ for $e \in E(\Gamma)$ and the trinion decomposition of Σ .

Thus a Riemann surface with a set $\{C_i\}$ is completely determined by a trivalent graph Γ , and we can denote it by the symbol Σ_Γ . We have the map

$$c_\Gamma: \text{ClRep}(\pi_1(\Sigma)) \rightarrow \mathbb{R}^{3g-3} \quad (2.7)$$

with fixed coordinates (c_1, \dots, c_{3g-3}) such that

$$c_i = c_{C_i}. \quad (2.8)$$

Then

- (1) c_Γ is a real polarization of the system $(\text{ClRep}(\pi_1(\Sigma)), k \cdot \Omega)$.
- (2) The coordinates c_i are action coordinates for this Hamiltonian system.
- (3) The map c_Γ is a *moment map* for the Hamiltonian action of T^{3g-3} on $\text{ClRep}(\pi_1(\Sigma))$

$$\text{ClRep}(\pi_1(\Sigma)) \times T^{3g-3} \rightarrow \text{ClRep}(\pi_1(\Sigma)) \quad (2.9)$$

described in [7].

- (4) The image of $\text{ClRep}(\pi_1(\Sigma))$ under c_Γ is a convex polyhedron

$$\Delta_\Gamma \subset [0, 1]^{3g-3}. \quad (2.10)$$

- (5) The symplectic volume of $\text{ClRep}(\pi_1(\Sigma))$ equals the Euclidean volume of Δ_Γ :

$$\int_{\text{ClRep}(\pi_1(\Sigma))} \Omega^{3g-3} = \text{Vol } \Delta_\Gamma = \frac{2 \cdot \zeta(2g-2)}{(2\pi)^{g-1}}.$$

These functions c_i are continuous on all $\text{ClRep}(\pi_1(\Sigma))$ and smooth over $(0, 1)$. Recall that a Hamiltonian torus action on $\text{ClRep}(\pi_1(\Sigma))$ is given by any closed trivalent graph Γ of genus g .

Summarizing we have

- (1) the convex polyhedron $\Delta_\Gamma \subset [0, 1]^{3g-3}$;

- (2) the part of the boundary

$$P_r = \partial\Delta_\Gamma \cap \partial[0, 1]^{3g-3} \subset \partial\Delta_\Gamma; \quad (2.11)$$

- (3) the part of the boundary of the convex polyhedron

$$P_K = c_\Gamma(K_g) \subset \Delta_\Gamma; \quad (2.12)$$

- (4) the open subset

$$\Delta_\Gamma^0 = \Delta_\Gamma \setminus (P_r \cup P_K) \subset [0, 1]^{3g-3}; \quad (2.13)$$

- (5) the open toric space

$$c_\Gamma^{-1}(\Delta_\Gamma^0) = \text{ClRep}(\pi_1(\Sigma))^0 \subset \text{ClRep}(\pi_1(\Sigma)) \quad (2.14)$$

relative compact with respect to the moment map

$$c_\Gamma: \text{ClRep}(\pi_1(\Sigma))^0 \rightarrow \Delta_\Gamma^0. \quad (2.15)$$

We will construct all of these things in the next section.

3 Combinatorial constructions

Here our basic set-up is from [6]. Recall that any trivalent graph Γ is given by the set of vertices $V(\Gamma)$ and the “incidence” quadratic form. Namely let $\mathbb{Z}^{V(\Gamma)}$ be the free \mathbb{Z} -module of all formal linear combinations of vertices with coefficients in \mathbb{Z} . Of course the set of vertices is a basis of this module. Let q_Γ be the symmetric matrix with entries

$$\alpha_{v_i, v_j} = \text{the number of edges joining vertices } v_i, v_j \in V(\Gamma).$$

Of course the group of permutation of $V(\Gamma)$ acts by permuting rows and columns.

Recall (see [6]) that a graph Γ is called *hyperbolic* if there are two subset $V_+, V_- \subset V(\Gamma)$ such that the spaces \mathbb{Z}^{V_\pm} are isotropic with respect to q_Γ . The matrix of a hyperbolic graph has the block form

$$q_\Gamma = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}, \quad (3.1)$$

where the blocks

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{V_+}, \mathbb{Z}^{V_-}) \quad (3.2)$$

give an identification

$$*: V_+ \leftrightarrow V_- \quad (3.3)$$

The set of edges of a hyperbolic graph $E(\Gamma)$ can be presented as the disjoint union of triples with common vertex

$$E(\Gamma) = \bigcup_{v \in V_+} E(\Gamma)_v \quad (3.4)$$

where $E(\Gamma)_v$ is the set of 3 edges from a vertex $v \in V_+$.

Now let Σ_Γ be the result of our graph pumping and consider the subset

$$\Sigma_+ = \bigcup_{v \in V_+} \tilde{v} \subset \Sigma_\Gamma = \bigcup_{v \in V_+ \cup V_- = V(\Gamma)} \tilde{v} \quad (3.5)$$

which is called a *half Riemann surface* Σ_Γ (see [6]).

All these constructions hold for any trivalent graph, not necessary connected. In particular, consider the disjoint union

$$\Theta^{g-1} = \Theta \sqcup \cdots \sqcup \Theta. \quad (3.6)$$

This trivalent graph of genus g determines the Riemann surface

$$\Sigma_{\Theta^{g-1}} = \Sigma_\Theta \sqcup \cdots \sqcup \Sigma_\Theta \quad (3.7)$$

which is the disjoint union of $g - 1$ copies of a Riemann surface of genus 2 with the standard trinion decomposition corresponding to the graph Θ .

We fix one vertex from the trinion decomposition of each copy of Σ_Θ , and denote this set of vertices by $V_+ \subset V(\Theta^{g-1})$ and its complement by V_- . They generate isotropic submodules with respect to $q_{\Theta^{g-1}}$. Thus the graph Θ^{g-1} is hyperbolic with the natural identification $*$ sending a trinion \tilde{v} with $v \in V_+$ to the second trinion of the component Σ_Θ .

Now the half Riemann surface Θ^{g-1} is

$$\Sigma_+ = \bigcup_{v \in V_+} \tilde{v} \subset \Sigma_{\Theta^{g-1}}, \quad (3.8)$$

which coincides precisely with the half Riemann surface Σ_Γ :

$$\Sigma_\Gamma \supset \Sigma_+ \subset \Sigma_{\Theta^{g-1}}. \quad (3.9)$$

4 Classes of representations spaces

The spaces $\text{ClRep}(\pi_1(\Sigma))$ and $(\text{ClRep}(\pi_1(\Sigma)))^{g-1}$ are symplectic spaces with toric structures defined by the graphs Γ and Θ^{g-1} (see Section 1), and with moment maps

$$c_\Gamma: \text{ClRep}(\pi_1(\Sigma)) \rightarrow \Delta_\Gamma \quad (4.1)$$

and

$$c_{\Theta^{g-1}}: (\text{ClRep}(\pi_1(\Sigma)))^{g-1} \rightarrow \Delta_{\Theta^{g-1}}. \quad (4.2)$$

Proposition 4.1 *The polyhedron $\Delta_{\Theta^{g-1}}$ is the direct product of $g-1$ copies of the tetrahedron Δ_{Θ}*

$$\Delta_{\Theta^{g-1}} = \prod_{v \in V_+} \Delta_{\Theta}. \quad (4.3)$$

We can say more. Let $\text{ClRep}(\pi_1(\Sigma_+))$ be the space of classes of $\text{SU}(2)$ -representations of the fundamental group of half of $\Sigma_{\Theta^{g-1}}$. This space admits the map

$$c_{\partial\Sigma_+} : \text{ClRep}(\pi_1(\Sigma_+)) \rightarrow \Delta_{\Theta^{g-1}}. \quad (4.4)$$

Proposition 4.2 *The map $c_{\partial\Sigma_+}$ is an isomorphism.*

The proof is the “direct product” of [7], Proposition 3.1.

Now the embedding $\Sigma_+ \hookrightarrow \Sigma_{\Gamma}$ induces the restriction map

$$r : \text{ClRep}(\pi_1(\Sigma)) \rightarrow \text{ClRep}(\pi_1(\Sigma_+)) \quad (4.5)$$

and the map c_{Γ} is the composite

$$c_{\Gamma} = r \circ c_{\partial\Sigma_+} \quad (4.6)$$

because $\partial\Sigma_+$ is precisely the collection $\{C_e\}$ for $e \in E(\Gamma)$. So we have

Proposition 4.3 *The polyhedron Δ_{Γ} is contained in the image of $c_{\partial\Sigma_+}$. Thus*

$$\Delta_{\Gamma} \subset (\Delta_{\Theta})^{g-1}. \quad (4.7)$$

Now it is easy to check the following well known statement (see for example [9], Proposition 3.3.5).

Proposition 4.4 *The polytope Δ_{Γ} is obtained by taking*

- (1) *the product of all tetrahedrons corresponding to trinions*
- (2) *with linear constraints given by equalities of gluing of two trinions.*

We get immediately

Corollary 4.5 (1) *The constraint (1) for Δ_Γ is equivalent to the same thing for $(\Delta_\Theta)^2$;*
 (2) *we must replace the gluing equality of constraints (2) by the corresponding inequalities.*

To describe the Delzant model, we must present the transformation from Δ_Γ to $(\Delta_\Theta)^{g-1}$ as an inductive procedure by compositions of elementary transformations of polyhedrons. We do this in the following sections.

5 Moment polyhedron manipulations

Consider first a special trivalent graph of genus g , the so-called *multi-theta graph* $g\Theta$ of [6], Figures 1, 2 and 3. This is a vertical oval O crossed by $g - 1$ horizontal strings

$$\{e_{g-1}, e_g, \dots, e_{2g-3}\}. \quad (5.1)$$

This graph is symmetric about the vertical axis a_0 , and we write

$$i_0: g\Theta \rightarrow g\Theta \quad (5.2)$$

for the reflection in this axis. There are $g - 1$ vertices

$$v_1, \dots, v_{g-1} \quad (5.3)$$

on the left side of the graph, numbered from top to bottom.

Let

$$V_+ = \{v_1, i_0(v_2), v_3, i_0(v_4), \dots\} \quad (5.4)$$

be the half of $V(g\Theta)$ and $V_- = i_0(V_+)$.

Then $g\Theta$ is hyperbolic (3.1) with the isotropic subspaces \mathbb{Z}^\pm and $* = i_0$ (3.3). From the shape of this graph we can see that there is the set of edges on the left side of the oval O

$$\{e_1, e_2, \dots, e_{g-2} \mid e_i = \partial(v_i) \cap \partial(i_0(v_{i+1}))\}. \quad (5.5)$$

Just from the shape of this graph we can see that only the $g - 2$ edges

$$\{e_1, e_2, \dots, e_{g-2}\} \quad (5.6)$$

give nontrivial combinatorial flips. Each such edge e_i determines a coordinate t_3^i of \mathbb{R}_i^3 and a coordinate t_3^{i+1} of \mathbb{R}_{i+1}^3 .

5.1 Case $g = 3$

In this case the set of horizontal strings (5.1) is $\{e_2, e_3\}$ and the set of vertices (5.3) is equal $\{v_1, v_2\}$. The subset (5.4) is equal

$$V_+ = \{v_1, i_0(v_2), v_3, i_0(v_4)\} \quad (5.7)$$

and (5.5) is

$$\{e_1\} \quad \text{with} \quad e_1 = \partial(v_1) \cap \partial(i_0(v_2)). \quad (5.8)$$

Now to describe the constraints (2) of Corollary 4.5, consider the following involutions of $\mathbb{R}^6 = \mathbb{R}_1^3 \times \mathbb{R}_2^3$:

(1) interchange of 3-spaces

$$i_{12}(\mathbb{R}_1^3) = \mathbb{R}_2^3; \quad (5.9)$$

(2) interchanging two coordinates from 3-spaces \mathbb{R}_1^3 and \mathbb{R}_2^3 :

$$i_{e_1}(t_3^1) = (t_3^2). \quad (5.10)$$

Recall that we already have the constraints (1) of Proposition 4.4:

$$|t_1^i - t_2^i| \leq t_3^i \leq t_1^i + t_2^i \quad \text{for} \quad i = 1, 2. \quad (5.11)$$

But now we have to glue trinions v_1 and v_2 along e_1 . It is easy to see that

Proposition 5.1 *The constraints (2) of Corollary 4.5 are equivalent to the conditions*

$$|t_1^i - t_2^i| \leq t_3^j \leq t_1^i + t_2^i \quad \text{for} \quad i \neq j. \quad (5.12)$$

From this we have immediately

Theorem 5.2 *The moment polytope is given by*

$$\Delta_{3\Theta} = (\Delta_\Theta)^2 \cap i_{e_1}((\Delta_\Theta)^2). \quad (5.13)$$

Indeed, the involution i_{12} preserves our polyhedron $(\Delta_\Theta)^2$. Thus (5.13) is the geometric interpretation of the inequalities (5.12).

Recall that the tetrahedron Δ_Θ is the convex hull of the set S of 4 points in \mathbb{R}^3 :

$$\Delta_\Theta = \langle (0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0) \rangle \quad (5.14)$$

Thus $(\Delta_\Theta)^2$ is the convex hull of the 16 points $S_1 \times S_2$ in $\mathbb{R}^6 = \mathbb{R}_1^3 \times \mathbb{R}_2^3$.

Proposition 5.3 *The polytope $\Delta_{3\Theta}$ is the convex hull of the 8 points²*

$$\{(*, *, 0, *, *, 0)\} \cup \{(*, *, 1, *, *, 1)\}. \quad (5.15)$$

Indeed, it is easy to see that $t_3^1 \neq t_3^2$ violates the inequalities (5.12).

The beautiful description of the situation comes from real algebraic geometry. Namely, let C be a real algebraic curve of genus $g = 2$ with real theta characteristics; its Kummer surface is a real quartic K_2 with 16 real nodes $\{p_1, \dots, p_{16}\}$ in the real \mathbb{CP}^3 . Near the real linear hull of this set of nodes, our \mathbb{CP}^3 is just $\mathbb{R}^6 = \mathbb{R}^3 \times i\mathbb{R}^3$. Then the convex hull

$$\langle p_1, \dots, p_{16} \rangle = \Delta_{\Theta^2} = \Delta_\Theta \times \Delta_\Theta \subset \mathbb{R}^6 \quad (5.16)$$

is the Delzant polyhedron of $(\mathbb{CP}^3)^3$ with the natural torus action. There are six lines through every vertex p_i with six vertices on each line, as in the classic Kummer configuration 16_6 . In these terms you can see 8 required vertices and the convex polyhedron $\Delta_{3\Theta}$. It is easy to make these polyhedra integral.

5.2 Induction over g

Our strategy in what follows is quite simple. From the combinatorial point of view we have a sequence of polyhedra as a sequence of approximations of the polyhedra $\Delta_{g\Theta}$:

- (1) the first approximation is $(\Delta_\Theta)^{g-1}$;
- (2) the second³ approximation is $(\Delta_\Theta)^{g-3} \times \Delta_{3\Theta}$;

²What is *? If you allow all choices $* = 0, 1$ you get 2^8 choices, which is more than 8.

³Sorry for the funny numbering $2 \mapsto 3$ and so on

- (3) the i th approximation is $(\Delta_\Theta)^{g-i} \times \Delta_{i\Theta}$;
- (4) the final $(g-1)$ st approximation is of course $\Delta_{g\Theta}$ itself.

Thus we can use induction on g . Remark that at the last step of induction, we have

- (1) the polyhedron

$$\Delta_\Theta \times \Delta_{(g-2)\Theta} \subset \mathbb{R}^3 \times \mathbb{R}^{3(g-2)} \quad (5.17)$$

corresponding to the disjoint union $\Theta \cup (g-2)\Theta$;

- (2) in the second component $\Delta_{(g-2)\Theta}$, the trinion v_l is distinguished by the previous inductive step as the lowest trinion of $\Delta_{(g-3)\Theta}$. Thus we have the decomposition

$$\mathbb{R}^{3(g-2)} = \mathbb{R}_l^3 \times \mathbb{R}^{3(g-3)}; \quad (5.18)$$

- (3) there are distinguished edges

$$e \in E(\Theta) \quad \text{and} \quad e \in E_{v_l}((g-3)\Theta) \quad (5.19)$$

along which we glue the Riemann surfaces Σ_Θ and $\Sigma_{(g-1)\Theta}$;

- (4) so we have distinguished coordinate axes

$$\text{the } t_3\text{-axis in } \mathbb{R}^3 \quad \text{and} \quad \text{the } t_3^l\text{-axis in } \mathbb{R}_l^3 \quad (5.20)$$

corresponding e between standard coordinates (t_1, t_2, t_3) in \mathbb{R}^3 and (t_1^l, t_2^l, t_3^l) in \mathbb{R}_l^3 .

Now our gluing constraints are exactly the same as (5.12): in the last notation

$$|t_1 - t_2| \leq t_3^l \leq t_1 + t_2, \quad (5.21)$$

$$|t_1^l - t_2^l| \leq t_3 \leq t_1^l + t_2^l. \quad (5.22)$$

By the same argument as in Proposition 5.1, we get

Proposition 5.4 *The polytope $\Delta_{g\Theta}$ is the convex hull of 2^g points*

$$\{(*, *, 0, *, *, 0, *, \dots, *)\} \cup \{(*, *, 1, *, *, 1, *, \dots, *)\} \subset \mathbb{R}^{3(g-1)}, \quad (5.23)$$

where the $*$ are any choice of 0 or 1.

A slightly different description of the moment polyhedron as a subpolyhedron of Δ_{Θ}^{g-1} was given by Florentino [10].

Recall (see for example [8]) that a complex polyhedron $\Delta \subset \mathbb{R}^n$ is *Delzant* if for every vertex v , there exists an $n \times n$ integral matrix A with determinant ± 1 such that the map

$$t \in \mathbb{R}^n \rightarrow At - v \quad (5.24)$$

sends a neighborhood of $v \in \Delta$ onto a neighborhood of zero in \mathbb{R}^n .

In particular a complex polyhedron $\Delta \subset \mathbb{R}^n$ is Delzant iff

- (1) (topological condition) its 1-skeleton (the union of edges) is an n -valent graph Γ .
- (2) The set $E(\Gamma)_v \subset E(\Gamma)$ of edges containing a vertex $v \in V(\Gamma)$ gives a rational basis in \mathbb{R}^n .

Of course a direct product of Delzant polyhedra is again Delzant.

Proposition 5.5 *The polyhedron $\Delta_{g\Theta} \subset \mathbb{R}^{3g-3}$ is Delzant.*

For the proof we can use induction on g . Actually it is enough to consider the case $g = 3$. But let us start with the case $g = 2$. Here we have the unit cube $C = [0, 1]^3$ with 8 vertices. To construct from it our tetrahedron Δ_2 we get the coordinate origin $(0, 0, 0)$ and choose all vertices on the distance $\sqrt{2}$. The convex hull of these 4 vertices is our tetrahedron Δ_2 . We carry out the same procedure for $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$: we choose all vertices of the unit cube in \mathbb{R}^6 at distance 2 from the origin and for each of these we get the same type set of vertices and so on. After this we take their convex hull. Finally, for genus g , the polyhedron $\Delta_{g\Theta} \subset \mathbb{R}^{3g-3}$ is the convex hull of vertices of the unit cube in \mathbb{R}^{3g-3} of distance $\sqrt{2g-2}$ from the origin and so on. We are done.

6 Delzant model

Now we have a precise description of the image of the moment map of the Hamiltonian torus action on $\text{ClRep}(\pi_1(\Sigma))$. It turns out that this polytope is Delzant. Thus by the main theorem of the Delzant theory we have the *smooth* Hodge manifold DM_g with a Hamiltonian action of T^{3g-3} .

Definition 6.1 *The smooth symplectic manifold DM_g is called the Delzant model of $\text{ClRep}(\pi_1(\Sigma))$ (or of $M_C(2, 0)$).*

The direct construction of this manifold is described in a lot of references but [8] is best.

The list of properties is following

- (1) The smooth algebraic variety DM_g has a canonical polarization H .
- (2) The dimension of $H^0(DM_g, H^k)$ can be computed in terms of $\frac{1}{2k}$ -integer points of $\Delta_{g\Theta}$ by the Duistermaat–Heckman formula as in [8], Chap. 3. This dimension is given by the Verlinde number, i.e., it is equal to the dimension of space of conformal blocks of level k and genus g .
- (3) The set of points

$$(\Delta_{g\Theta})_{2k} = \frac{1}{2k} \mathbb{Z}^{3g-3} \cap \Delta_{g\Theta} = BS_k \quad (6.1)$$

is the set of Bohr–Sommerfeld fibers of the fibration $c_{\Delta_{g\Theta}}$ (2.7) of level k .

The (symplectic) geometric picture is given by the two fibrations over the same base:

$$\text{ClRep}(\pi_1(\Sigma)) \xrightarrow{c_{\Delta_{g\Theta}}} \Delta_{g\Theta} \xleftarrow{m} DM_g \quad (6.2)$$

where m is the moment map of the torus manifolds.

6.1 Comparison with “mirror fibrations”

The typical (conjectural) set-up of the SYZ-mirror construction [11] also consists of two dual Lagrangian fibration over the same base. We can view both fibrations as families of Lagrangian cycles with degenerations.

The right hand family

$$\mathrm{ClRep}(\pi_1(\Sigma)) \xrightarrow{c_{\Delta_{g\Theta}}} \Delta_{g\Theta}$$

is an equidimensional family with singular fibers.

The left hand family

$$DM_g \xrightarrow{m} \Delta_{g\Theta}$$

has fibers i -tori T^i . Namely let $\mathrm{sk}_i(\Delta_{g\Theta})$ be the i -skeleton of $\partial\Delta_{g\Theta}$. Then

$$p \in \mathrm{sk}_i(\Delta_{g\Theta}) \setminus \mathrm{sk}_{i+1}(\Delta_{g\Theta}) \implies m^{-1}(p) = T^i \quad (6.3)$$

is an i -dimensional torus. Moreover every i -dimensional face F_i defines a projective subspace $\mathbb{P}^i(F_i) \subset DM_g$ with an i -torus action which is a Delzant space. Thus in the Delzant model DM_g we have the configuration of projective subspaces corresponding to drop in fiber dimensions. This is the typical behavior for the isotropic fibers of a prequantized completely integrable dynamical system.

7 Conformal blocks

We saw that for any complex curve Σ , the two compact complex polarized varieties $M_\Sigma(2, 0)$, Θ and DM_g, H admit equidimensional spaces of conformal blocks of level k

$$H^0(M_\Sigma(2, 0), \Theta^k) \quad \text{and} \quad H^0(DM_g, H^k). \quad (7.1)$$

We use the following statement to relate these spaces canonically.

Proposition 7.1 *The polyhedron $\Delta_{g\Theta}$ admits a unique internal barycenter c_0 of symmetry.*

Near the fibers

$$c_{g\Theta}^{-1}(c_0) \quad \text{and} \quad m^{-1}(c_0) \quad (7.2)$$

we can identify our toric spaces using equivariant Darboux–Weinstein coordinates. In particular we identify the fibers

$$c_{g\Theta}^{-1}(c_0) = m^{-1}(c_0) = T^{3g-3}. \quad (7.3)$$

Both of these tori are Lagrangian so that the restrictions $\Theta|_{c_g^{-1}(\Theta)}^{-1}(c_0)$ and $H|_{m^{-1}(c_0)}$ are trivial line bundles with flat connections which are gauge equivalent. The equivariant Darboux–Weinstein lemma can be extended to identification of the line bundles with unitary connections under the identification (7.3).

Summarizing, we have the torus T_0^{3g-3} equipped with the trivial line bundle (L_0, a_0) with a flat connection and the Lagrangian embeddings

$$\mathrm{ClRep}(\pi_1(\Sigma)) \supset c_{g\Theta}^{-1}(c_0) \hookleftarrow T_0^{3g-3} \hookrightarrow m^{-1}(c_0) \subset DM_g \quad (7.4)$$

such that the preimages of Θ and H are equal to (L, a) .

Then the restriction maps

$$H^0(M_\Sigma(2, 0), \Theta^k) \rightarrow \Gamma^\infty(L_0) \leftarrow H^0(DM_g, H^k) \quad (7.5)$$

are embeddings and give the identification of spaces (7.1).

Thus around nonsingular points of $\mathrm{ClRep}(\pi_1(\Sigma))$ with a smooth torus action this space is modelled by the linear actions of tori on complex projective spaces as predicted by the equivariant (Darboux)–Weinstein theorem. For singular points we have to find new local model instead of \mathbb{CP}^n . We will do this in a subsequent paper.

8 Acknowledgments

The relations between non-Abelian theta functions and the Delzant theorem were raised by my collaborators J. Mourao, J. P. Nunes and C. Florentino. I would like to express my gratitude to all of them, to Bill Oxbury and specially to Miles Reid who shown me a gap in the constrain (1.3). I would also like to thank the Korea Institute for Advanced Study (KIAS, Seoul) for support and hospitality.

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